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NOTES ON COMPLETENESS OF THE INTUITIONISTIC PREDICATE CALCULUS

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0. Notations

(n) refers to a remark, [n] - to a reference.

0.1. L_P - the language of pure predicate calculus consists of

(1) formal letters: P_i^j, a_i (i,j=0,1,2,...) for j-place predicates and for parameters (respectively)

(2) logical symbols: &, $\vee$, $\rightarrow$, $\neg$, $\wedge$ (for absurdity), $\exists$, $\forall$.

(3) auxiliary symbols: (,), x, x_i (i=0,1,...) etc.

$\hat{L}_P$ is the class of wff of L_P, as defined e.g. in [Prawitz 65].

L_{Pf} enlarges L_P by the addition of formal letters f_i^j (i,j=0,1,2,...) for j-place functions.

$\hat{L}_{Pf}$ includes accordingly some additional rules of formation.

L_A enlarges L_P by the addition of numerals.

$\hat{L}_A$ - accordingly.

L_{Af} is the union of L_A and L_{Pf}.

$\hat{L}_{Af}$ - accordingly.

0.2. I denotes the intuitionistic first-order predicate calculus, i.e.: the class $I \subset \hat{L}_P$ generated by some formal system of axioms and rules of inference for intuitionistic first-order logic.

C denotes the classical first-order predicate calculus.

If $\subset \hat{L}_{Pf}$ and Cf - accordingly.

IA $\subset \hat{L}_A$ denotes intuitionistic, first-order arithmetics, i.e. - Heyting's arithmetic.

CA denotes classical first-order arithmetics, as generated e.g. by adding to IA the rule of excluded-middle.

IAf and CAf - accordingly.

0.3. In metalanguage we use formal letters as their own names.

A, B_i (i=0,1,...) etc. stands for formulas, $\underline{A}$, $\underline{B}$ etc. - for occurrences of formulas.

A & B, A ∨ B etc. are obvious (non-precise) abbreviations.

$\bigwedge_{j \leq i \leq k} A_i$ stands for multiple conjunction.

The meanings of <u>and</u>, <u>or</u>, $\Longrightarrow$, ~, ∃, ∀ (as metamathematical symbols) are obvious; $\asymp$ stands for (metamathematical) equality and we use the usual set-theoretic notation.

$A(\frac{\underline{B}}{\underline{C}})$ is the formula which results by substituting $\underline{C}$ in place of $\underline{B}$ in A

If β and γ are sequences of occurrences of formulas, of the same length, $A(\frac{\beta}{\gamma})$ is defined inductively in an obvious way.

0.4. If T is some theory (e.g.: I. or CAf), then $\vdash_T$ stands for provability in T,

$\mathbb{P}rov_T(x,A)$ means that x is a proof of A in T.

$\ulcorner A \urcorner$ denotes the Gödel's number of A in some fixed enumeration.

$Prov_T(x,\ulcorner A \urcorner)$ is the number-theoretical formalized form of $\mathbb{P}rov_T(x,A)$, i.e.: x is the Gödel-number of a proof of A in T.

M denotes some model (cf. 2.2).

A^M is the interpretation of A in M.

$M \models A$ denotes satisfaction.

If a model M of T is described in the language L, and $K \subset \hat{L}$, then $M \subset K$ means $\{A^M \mid A \in \hat{L}_T\} \subset K$.

A_S stands for the Skölem-normal-form of A.

I. Metamathematical setting

1. Intuitionistic validity

1.0. Validity of a formula is intended to mean roughly "satisfaction in any situation". The three terms employed correspond to three mental constructions:

1.0.1. "A mathematical situation" (a "model") means an interpretation of the formal letters of the language. For L_P these are

(i) The parameters, which are interpreted as a plurality, the universe of discourse.

(ii) The predicates. (1)

1.0.2. "Satisfaction" is defined inductively over logical complexity, and is thus redecud to an interpretation of the logical symbols (connectives and quantifiers).

1.0.3. "Any" involves a notion of exhaustive variation over arbitrary models.

1.1. For C (the classical first-order predicate calculus) the definition of validity is effected with the leading principles of apriority and maximality, that is:

1.1.1. (i) "Plurality" is meant to be

(i1) a given, completed set D

(i2) in which every element is independent. (2)

(ii) An n-place predicate is interpreted as a subset S of D^n, conceived in the same way.

1.1.2. Logical symbols are interpreted as well-defined and complete truth-functions.

1.1.3. A model of a formula is thus a special definable kind of set, and the class of models can in turn become the domain of variation of a universal quantifier (of higher order).

1.2. An intuitionistic notion of validity for first-order formulas must be constructed with some cautions.

1.2.1. Formal letters are interpreted by species, and therefore neither are they delimited pluralities, nor their elements independent.

1.2.2. In contrast to the classical case, where logical symbols are interpreted set-theoretically by two-valued truth functions, in intuitionistic mathematics there is no universal predicative interpretation of logical symbols. Every assertion, e.g. $A \to B$ or $\forall x\, A(a,x)$ is interpreted in a structure as a construction in the form of a functional of higher order, which results from a genuine inspection on the structure, or which follows by a logical inference from such inspections. It is not given beforehand by the interpretation of the formal letters.

1.2.3. Given a structure M and a formula A, $M \models A$ is asserted therefore only by a "proof". If many formulas are considered, we can speak of a "theory". We must understand here "theory" in the most general sense; not as a formal theory generated by a formal system, but as a set of formulas shown to hold in the structure by any mean. The construction in the proof of Gödel's incompleteness theorem provides an evident illustration of the difference between the two. Hence we have on the one hand the parallel of the classical notion of a model - a description by species of the domain and the predicates, and on the other hand a theory which refers to it. Contrary to classical metamathematics, where "true formulas in the model" are assumed to be defined à priori, and "true formulas" in general to be defined in a "closed" (set-theoretic) semantics, we have here an open concept of truth, which corresponds to intuitionistic view of mathematics in general.

1.2.4. By this interpretation we have

$$(*)\ M \not\models A \Longrightarrow M \models \neg A$$

because the antecedent means that it can be shown by some metamathematical argument that $M \models A$ is absurd. By 1.2.3 such an argument is accepted for the "theory" of M, and therefore $M \models \neg A$.
However, one does not have

$$M \models A \quad \underline{\text{or}} \quad M \not\models A.$$

1.3. Could we incorporate the general notion of "model + knowledge" into universal intuitionistic theory, as classical semantics is described, in set-theorem (3)?

We have seen that the theory of species does not provide such a general basis (4). The description of the model by species does not contain the "theory" of the model, because when speaking about the species, about "equality", "inclusion", "distinction" etc., one uses the terms "to construct" and "to establish" without specifying their meaning. Thus, in the theory of species a certain aspect of the process of construction of mathematics is reflected, but its existence presupposes this construction.

This emphasizes explicitly the argument of Brouwer that no set theory can serve as the basis of mathematics. Further, any attempt to have a closed universal description of semantics by some modified theory of species is contrary to this view of mathematics as an always-open investigation.

1.4. Quantification over models

1.4.1. A notion of validity involves a form of quantification $\forall M$ ... where M ranges over models.

This can be understood as: "regardless of the model, ...". In this case one disregards the exact meaning of concepts like "model" and "semantics" in general, and "..." is asserted on a logical basis (5). However, if we do assert such a thing, we assume it to be written-out already in the formal system we consider, and the problem of completeness is precisely to check whether some new formula is valid by actual semantic considerations.

In order to present a sensitive problem of completeness one has therefore to examine concretely the meaning of "$\forall M$ ...".

1.4.2. The first problem raisen in this is the general problem of quantification over a kind of objects in generation, like kinds of free-choice sequences. Every model is in generation, at least the "theory" of the model is, and the quantification makes sense when we assert the matrix of the universal quantifier on the base of an "initial segment" of its range. One would like, for instance, to have some bound on the complexity of "proofs" needed to assert for arbitrary M that $M \models A$.

1.4.3. But the main complication lies in the range of the quantifier, i.e. the "totality of models" itself. To see this we should keep in mind that we do not look for a general logical scheme, but on some formula asserted really by semantic considerations. Following 1.3 we are forced to reject the unlimited universal quantification altogether (6).
A meaningful solution for the problem of completeness can be, therefore, only a strong solution, on the ground of limited semantics (e.g.: intuitionistic analysis of a certain order). This is not necessarily an essential handicap, since the completeness of C (the classical predicate calculus) was proved in classical metamathematics by Hilbert-Bernays [39] on the ground of a very restricted semantic.

1.4.4. The same situation arises when one likes to assert invalidity. The universal quantification is again the antecedent of an implication, and the non-validity can be concluded only in a strong form, on the basis of restricted semantic.

1.4.5. The full meaning of this interpretation one has to give to validity in intuitionistic metamathematics appears when one tries to prove incompleteness. Here the universal quantification over models occurs positively. In principle, it cannot be asserted definitely about an unproved formula that it is valid, since we have rejected the concept of exhaustive validity.
However, incompleteness may be established if one add new metasemantic principles. More specifically: if the interpretation of the formal letters is restricted (e.g., if function-symbols are interpreted as recursive functions), then completeness may conflict with known results about the restricted interpretation. In this sense we have a negative result in §4, where, besides Church's thesis, we have the principle of 4.2.

1.4.6. Since positive completeness results are difficult to be obtained one would like to strengthen the notion of validity in some way, as a counterpart to 1.4.3. A step in this direction is the concept of "schematic validity" (Kreisel [58]). To obtain it one allows the species representing the formal symbols to depend on parameters. These become significant when they represent uncompleted objects, like lawless sequences. In this case, in fact, one cannot fix the parameters and get a well-defined model; nevertheless the concept of validity, resulting by universal quantification over the parameters, is well-defined. The passage is analogous to that between lawless sequences and quantification over them (7).

1.5. If one assumes that a model M of a theory T (in the language L_T) is describable in some language L, then one would expect the following:

(1) The formal letters of L_T are interpreted by uniformly-recursively defined combinations of L.

(2) For the types of formal-letters of L, employed in this description, L contains also proper names (constants).

(3) The interpreting expressions of L combines like the represented formal letters of L_T (e.g.: the representation of a (1)0 functional acts as such).

1.5.1. One may of course start with a formal theory T^* in L as "the theory of the model", but one must keep in mind the possibility to enrich T^* by metamathematical observations (cf. 1.2.3).

2. Kinds of completeness

2.1. Classical assertion of semantic completeness reads

(i) $\forall M \quad M \models A \Longrightarrow \exists x \mathrm{Prov}(x,A)$.

Already "formally" (i) splits into several variants, all equivalent classically, but not intuitionistically.
The disjunctive form

(ii) $\sim\forall M \quad M \models A$ or $\exists x \mathrm{Prov}(x,A)$

is of no interest, since the intuitionistic interpretation of disjunction implies here a decision of I, which is absurd.

A weaker variant is the double-negation of (i), which is equivalent to the three forms

(iii-1) $\forall M \quad M \models A \Longrightarrow \sim\sim \exists x\ \mathrm{Prov}(x,A)$

(iii-2) $\sim\exists x\ \mathrm{Prov}(x,A) \Longrightarrow \sim\forall M \quad M \models A$

(iii-3) $\sim[\forall M \quad M \models A$ <u>and</u> $\sim\exists x\ \mathrm{Prov}(x,A)]$.

Other forms results from intuitionistic interpretation of quantifiers:

(iv) $\sim\exists x\ \mathrm{Prov}(x,A) \Longrightarrow \exists M \quad M \models \neg A$

(v) $\sim\exists x\ \mathrm{Prov}(x,\neg A) \Longrightarrow \exists M \quad M \models A$

(iv) does not imply (v) trivially, because

$$M \models \neg\neg A \Longrightarrow M \models A \quad \text{does not hold.}$$

We call these forms (8)

(i) Strong completeness
(iii) Weak completeness
(iv) Counter-example (CE-) completeness
(v) Satisfiability of the irrefutable.

2.1.1. Although in the case of weak completness the effectiveness of implication is abandoned, establishing weak-completeness is amply sufficient for the intended purpose of completeness (9).

2.1.2. The interes is further focussed to weak completeness by Gödel's discovery reported in Kreisel [62]: strong completeness (for the negative fragment) implies

(*) $\neg\neg\exists n\ An \to \exists n\ An$ for every p.r. predicate A.

Interest in stronf completeness reduces consequently to weak completeness and (*). If (*) holds then, since Prov is a p.r. predicate,weak completeness implies strong completeness; while if (*) is refuted, then so is strong completeness of I.

2.2. Other variants are of minor or no interest, as a consequence of the following observations:

2.2.1. The metamathematicsl predicate Prov_T is intended to be decidable for any intuitionistic theory T. Therefore, variants in which $\exists x\, \mathbb{P}\mathrm{rov}.(x,A)$ is replaced by $\exists x \sim\sim \mathbb{P}\mathrm{rov}(x,A)$ do not have to be considered.

2.2.2. The limited interest in strong completeness, following 2.1.1 and 2.1.2, reduces the interest in variants stronger than strong-completeness, in which a weaker form of validity $\sim\sim \forall M \quad M \models A$ appears (not to mention difficulties of interpretation, like those of 1.4.4).

2.2.3. The interest in weaker validity $\forall M \sim\sim M \models A$ is completely eliminated by the equivalence of this form (1.2.4) to $\forall M \quad M \models \neg\neg A$. A completeness result using this form would lead to the absurd implication

$$\exists x\, \mathbb{P}\mathrm{rov}(x,\neg\neg A) \Longrightarrow \text{(soundness)}$$

$$\forall M \quad M \models - - A \Longrightarrow$$

$$\Longrightarrow \forall M \sim\sim M \models A$$

(by the assumed completeness)$\Rightarrow$ $\exists x\, \mathbb{P}\mathrm{rov}(x,A)$.

2.2.4. If one takes in the provability-side of some formulation of completeness some classical-variant of A, e.g. $-\neg\neg A$, one does not get the intended completeness result for the whole of I.

2.3. Considering incompleteness, we should distinguish between a strong (local) form, by which a formula conflicting completeness can actually be constructed,

(i) $\exists A\{[\forall M \quad M \models A]$ <u>and</u> $\sim \exists x\, \mathbb{P}\mathrm{rov}(x,A)\}$,

and a weaker (global) form,

(ii) $\sim\forall A \sim \{[\forall M \quad M \models A]$ <u>and</u> $\sim \exists x\, \mathbb{P}\mathrm{rov}(x,A)\}$.

The result of §4 below is of the second kind.

II. Technical results

3. Completeness of fragments of I

3.0. By a completeness-result for a fragment $F \subset I$ we mean one for which the range of the parameter A in 2.1 is F. Prov_I is not modified to Prov_F.

3.1. Let $\phi: \hat{L}_P \mapsto \hat{L}_P$ satisfy the following conditions:

(i) $\vdash_{CA} \exists x\, \mathrm{Prov}_C(x, \ulcorner A \urcorner) \to \exists y\, \mathrm{Prov}_I(y, \ulcorner \phi A \urcorner)$.
Hence $\phi G \subset I$ and ϕ is a translation of C into I.

(ii) $\vdash_C A \to \phi A$

(iii) $\vdash_I \phi\wedge \to \wedge$

(iv) $\vdash_I \phi[A \to B] \to (\phi A \to \phi B)$

(v) ϕ is structural, that is:

$\phi[A(\frac{B}{P})]\ (\frac{P}{B}) \asymp \phi A$ where P is some predicate-letter not appearing in A.

Conclusions:

3.1.1. By (ii) and (iv): (vi) $\vdash_I \phi[\neg A] \to \neg\phi A$.

$\vdash_C \phi A \to \phi[\neg\neg A]$ (by (ii) and (iv))

$\to \neg\phi[\neg A]$ (by (vi))

$\to \neg\neg A$ (by (ii))

Hence $\vdash_C \phi A \to A$, and by (ii)

(vii) $\vdash_C \phi A \longleftrightarrow A$

whence also

(viii) $\vdash_I \phi A \longleftrightarrow \phi\phi A$.

3.1.2. ϕ can be naturally extended to

$$\bar{\phi}: \hat{L}_A \longmapsto \hat{L}_A \quad \text{in the following way.}$$

$P_i^j \in L_P$ can be interpreted primitive-recursively to enumerate all equations. As a result we get a translation (*) of $\hat{L}_A$ into $\hat{L}_P$. If for $A \in \hat{L}_P$ we have via this translation $\vdash_{CA} A^*$ then

$$\vdash_C A_0 \to (A_1 \to (A_2 \to \ldots (A_n \to A)))\ldots)$$

where $A_0^*, \ldots, A_n^*$ is the list of instances of axioms of CA used in the proof of A^*.
By (vii) $\vdash_C \phi A_0 \to (\phi A_1 \to \ldots (\phi A_n \to \phi A))\ldots)$

and since $A_0^*, \ldots, A_n^*$ are also axioms of IA

$$\vdash_{IA} [\phi A]^* \quad \underline{\text{or}} \quad \vdash_{IA} \bar{\phi} A^*$$

(ϕ is structural, therefore $[\phi A]^* \asymp \bar{\phi} A^*$)
and in fact

$$\vdash_{CA} \exists x \, Prov_{CA}(x, \ulcorner A^* \urcorner) \to \exists y \, Prov_{IA}(y, \ulcorner \bar{\phi} A^* \urcorner).$$

Conditions (ii)-(viii) are transferred from ϕ to $\bar{\phi}$ trivially.

3.1.3. By (vii) every $M \subset \hat{L}_A$ (cf. §0.4) is classically-equivalent to an $M^0 \subset \bar{\phi}\hat{L}_A$.

3.2. Theorem (Kreisel [58]): Let $\phi: \hat{L}_P \longmapsto \hat{L}_P$ satisfy conditions (i) - (v) of 4.1, and be such that $\sim \exists x \, \mathbb{P}rov(x, \phi A)$ is expressible in $\bar{\phi}\hat{L}_A$, for every $A \in \hat{L}_P$, by - say - R^A. Then ϕC is CE-complete.

Proof: In CA

$$R^A \longrightarrow \neg \exists x \, Prov_I(x, \ulcorner \phi A \urcorner)$$

$\longrightarrow \neg \exists x \, Prov_C(x, \ulcorner A \urcorner)$ (by (i))

$\longrightarrow \neg \exists x \, Prov_C(x, \ulcorner A_s \urcorner)$ (cf. 0.4)

$\longrightarrow \neg A_s^M$ (by Hilbert-Bernays [39], pp. 243-253, and by 4.1.3 assume $M \subset \bar{\phi}\hat{L}_A$)

$$\to \neg A^M$$

$$\to \neg \bar{\phi}A^M \qquad \text{(by (vii))}$$

Hence: (*) $\vdash_{CA} R^A \to \neg \bar{\phi}A^M$

and by (i), (iv) $\vdash_{IA} \bar{\phi}R^A \to \bar{\phi} \neg \bar{\phi}A^M$.

Since $R_A \in \hat{\bar{\phi}}_A$ this yields by (viii)

$$\vdash_{IA} R^A \to \neg \bar{\phi}[A^M]$$

and by (v), using $M \subset \hat{\mathcal{L}}_A$

$$\vdash_{IA} R^A \to [\neg\phi A]^M$$

To justify the proof in intuitionistic metamathematics, note that by the nature of the proof of (*) one has in fact for some n

$$\mathrm{Prov}_{CA}(n, \ulcorner R^A \to \neg\bar{\phi}A^M \urcorner)$$

and by full use of (i) the following steps provides p.r. instructions to compute some n_0 s.t.

$$\mathrm{Prov}_{IA}(n_0, \ulcorner R^A \to [\neg\phi A]^M \urcorner).$$

3.2.1. The Hilbert-Bernays construction still holds in exactly the same way for $\hat{\mathcal{L}}_{Pf}$, i.e.: a uniform construction yields for every $A \in \hat{\mathcal{L}}_{Pf}$ an M s.t.

$$\vdash_{IAf} \neg \exists x\, \mathrm{Prov}_{If}(x, \ulcorner \phi A \urcorner) \to \neg[\phi A]^M,$$

where M is a primitive-recursively defined arithmetical model of If, that is: predicate-letters are interpreted as arithmetical relations, and function-letters as p.r. functions.

3.3. In fact the description of M for every $A \in \hat{L}_{Pf}$ involves only the interpretation of the formal symbols (predicates and functions) actually appearing in A. Since these are p.r. defined and finite in number, we can arithmetize by a kind of Gödel-enumeration the series of models constructed.

The full power of the result above appears then in the form (in obvious shorten notation)

$$(*) \qquad \vdash_{IA} \forall \ulcorner A \urcorner \, \exists \ulcorner M \urcorner \, \{\exists x \, \mathrm{Prov}_{IA} \, (x, \ulcorner R^A \to [\neg\phi A]^M \urcorner)\}$$

where R^A express intuitionistically in $\bar{\phi}\hat{L}_{Af}$ the improvability of ϕA in IAf, $A \in \hat{L}_{Pf}$ and $\phi: \hat{L}_{PF} \mapsto \hat{L}_{Pf}$ as above.

It is legitimate to write here $\ulcorner R^A \to [\neg\phi A]^{M} \urcorner$ since

(i) ϕ is p.r., therefore

$$\ulcorner A \urcorner \longmapsto \ulcorner \neg\phi A \urcorner \quad \text{is p.r.}$$

(ii) exactly like the procedure for obtaining Gödel's substitution functions, one has that

$$\langle \ulcorner M \urcorner, \ulcorner \neg\phi A \urcorner \rangle \longmapsto \ulcorner [\neg\phi A]^{M} \urcorner \quad \text{is p.r.}$$

3.4. Examples

3.4.1. (cf. Leivant [71] for notation and an outline of proof). Define a translation ϕ by

$$\phi A \asymp_{Df.} \neg\neg A(\neg\neg^{\beta}\beta),$$

where β is the complete list of $\underline{B_i} \in S_A$ with the following properties:

(i) $\underline{\forall x_i \, B_i} \in S_A$

(ii) $\underline{B_i}$ is in the range of a disjunction or an existential quantifier.

(iii) $S^{+}_{B_i}$ does not have a clear bar free of x_i.

(iv) If $B_j < B_i$ with B_j satisfying (i) - (iii), then some $\vee$ or $\exists$ occurs between B_i and B_j in S_A.

3.4.2. (special case of 3.4.1). Define ϕ by $\phi A \asymp_{Df} \neg\neg A(\neg\overset{\beta}{\neg}\beta)$ where β is the complete list of $\underline{B_i}$ s.t. $\underline{\forall x_i B_i}$ is a subformula of A.
3.4.3. Gödel's translation (derived from 3.4.2. cf. Leivant [71] §7).
3.4.4. (subcase of 3.4.2): The class of formulas of the form $\neg\forall x_1 \ldots \forall x_k \neg A$ (with A open).

3.5. Invariance under negation of $\phi\hat{L}_P$ is necessary for the proof above (3.2): The class of negations of prenex formulas is the image of a translation satisfying all the conditions of 3.2 except (v) (cf. Kreisel [58] or Leivant [71] §9), but an elegant argument of Kreisel [58] (th. 6), shows that for such a formula A

$$\neg \exists x \, Prov_I(x, \ulcorner A \urcorner) \rightarrow \neg A^M$$

cannot be proved in IA for any arithmetical model M.

3.6. A simple argument (due to Kreisel [61] pp. 13-14) shows that example 3.4.4 can be generalized to obtain CE-completeness (relative to arithmetical models and proved within IA) for the more general class of formulas of the form $\neg Q \neg A$, where Q is a "block" of consecutive quantifiers and A is open.
For $A \in \hat{L}_A$, if $\vdash_{IA} \forall x \exists y \, A(x,y)$, then one has p.r. instructions to get for any given number n a number m s.t. $\vdash_{IA} A(n,m)$. Hence $\vdash_{IAf} \forall x \, A(x,fx)$ with a p.r. f. More generally:
Since IAf is a conservative extention of IA, one has that

$$\vdash_{IA} \neg\forall x_1 \exists y_1 \ldots \forall x_k \exists y_k \neg A(x_1,\ldots,x_k,y_1,\ldots,y_k)$$

implies

$$\vdash_{IAf} \neg\forall x_1 \ldots \forall x_k \neg A(x_1,\ldots,x_k,f_1 x_1,\ldots,f_k x_1 \ldots x_k).$$

By 3.4.4 then, there is a p.r.-defined arithmetical model M s.t.

$$\vdash_{IAf} \neg\neg\forall x_1 \ldots \forall x_k \neg[A(x_1,\ldots,x_k,f_1x_1,\ldots,f_kx_1\ldots x_k)]^M$$

and hence even

$$\vdash_{IA} \forall x_1\exists y_1 \ldots \forall x_k\exists y_k \neg[A(x_1,\ldots,x_k,y_1,\ldots,y_k)]^M.$$

3.6.1. The same method can be combined with 3.4.1 (instead of 3.4.4) to give CE-completeness of a larger class:that of stable formulas A (i.e., s.t. $\neg\neg A\to A$), for which every block of quantifiers has a stable matrix:
In fact it is enough to assume that only the blocks Q_i with the following properties have a stable matrix

(i) Q_i contains a universal quantifier.

(ii) Q_i is in the range of an existential quantifier.

(iii) If $Q_i < Q_j$ (in the natural order of S_A) with Q_i satisfying (i)-(ii), then some $\vee$ or $\exists$ occurs between Q_i and Q_j in S_A.

3.7. Schematic strong completeness relative to a kind of free-choice-sequences is established for the class of prenex-formulas of $\mathcal{L}_P$ in Kreisel [58a] §7. The weak completeness shown there gives strong-completeness, since this class is decidable (Rasiowa-Sikorski [54]), whence

$$\vdash_{IA} \neg\neg\exists x\, Prov_I(x,\ulcorner A\urcorner) \to \exists x\, Prov_I(x,A)$$

for every prenex A.
A subcase of this is the intuitionistic propositional calculus.

4. Incompleteness under Church's thesis

4.0. The following is a negative result of the kind mentioned in 1.4.5. It was announced in Kreisel [70]. We use a result in recursion theory due to Jockusch [70], instead of that of Mostowski proposed in Kreisel [70].

4.1. The essence of the argument is roughly this: the species of valid (in recursive interpretation of function-symbols) formulas $V \subset \hat{L}_{Pf}$ is of great complexity: it is not r.e. (in fact - it is $\sum_2^0$-complete). Hence V cannot be equal to the species of provable formulas, which is r.e.

The complexity of V is due to the power of expression implicite in the metamathematical assertion $A \in V$ $(A \in \hat{L}_{Pf})$. Such an assertion enables one to express the (non-) existence of a recursive i.p.s. in any p.r. describable species of i.p.s., and thus a certain form of separation between recursive and non-recursive i.p.s.

Instead of V, one can use in this argument $V \cap F$ with F any r.e. species of formulas. We take F to be the class of formulas asserting that the projection of the function 'f' in some p.r. binary tree is not infinite. This is a structural definition, and hence F is in fact p.r.

Assuming Church's thesis $V \cap F$ is then the class of formulas asserting that some p.r. binary tree has no infinite recursive branch. A lemma from recursion theory shows this class to be non-r.e. in a constructive way, which concludes the proof.

4.2. Consider the conjunction Z of the closure of the following formulas ("axioms") for the predicate-symbols P_1^2, P_1^1, P_2^2 which we designate by E, Z, S:

$$E(x,x), \quad E(x,y) \ \& \ E(x,z) \rightarrow E(y,z)$$

$$E(x,y) \ \& \ Z(x) \rightarrow Z(y); \quad Z(x) \ \& \ Z(y) \rightarrow E(x,y)$$

$$E(x,y) \ \& \ S(x,z) \rightarrow S(y,z); \quad E(x,y) \ \& \ S(z,x) \rightarrow S(z,y)$$

$$S(x,y) \ \& \ S(x,z) \rightarrow E(y,z)$$

$$\exists x Z(x); \quad \forall x \exists y S(x,y)$$

$$\neg[S(x,y) \& Z(y)].$$

We assume that every model of Z contains a constructively - specified submodel which is constructively isomorphic to the natural numbers, with E interpreted as equality, Z as zero and S as successor relations.

4.3. Let T be a p.r. binary tree, i.e. its characteristic function τ, for which

$$Tx \Longleftrightarrow \tau x = 0$$

has a p.r. definition.
Hence, there is a sequence of functions $\tau_0,\ldots,\tau_n$ s.t. τ_i $(0\le i\le n)$ is

either (1) $\tau_i(n) = 0$

or (2) $\tau_i(n) = n+1$

or (3) $\tau_i(n_1,\ldots,n_{r_i}) = n_q \quad (1 \le q \le r_i)$

or (4) $\tau_i(n_1,\ldots,n_{r_i}) \quad \tau_j(\tau_{k_1}(n_1,\ldots,n_{r_{k_1}}),\ldots,\tau_{k_l}(n_1,\ldots,n_{r_{k_l}}))$

with $j,k_1,\ldots,k_l < i$

or (5) $\begin{cases} \tau_i(m,0) = \tau_j(m) \\ \tau_i(m,n+1) = \tau_k(m,n,\tau_i(m,n)) \qquad \text{with } j,k < i \end{cases}$

and $\tau \asymp \tau_n$.
Assuming the intended interpretation of E, Z and S we can, using these predicate-symbols, define T by a mimic of the definition of τ:
Let F_i $(0\le i\le n)$ be one of the following, according to the case for the definition of τ_i above.

(1) $Z(y) \rightarrow P_i(x,y)$

or (2) $S(x,y) \rightarrow P_i(x,y)$

or (3) $P_i(x_1,\ldots,x_r,x_j) \qquad (1 \le q \le r)$

or (4) $\bigwedge_{1\le h\le l} P_{k_h}(x_1,\ldots,x_t,y_h) \;\&\; P_j(y_1,\ldots,y_l,y) \rightarrow P_i(x_1,\ldots,x_r,y)$

or (5) $[Z(z) \;\&\; P_j(x,y) \rightarrow P_i(x,z,y)] \;\&$

$[P_i(x,y,z) \;\&\; S(y,k) \;\&\; P_k(x,y,z,v) \rightarrow P_i(x,u,v)]$.

Let now $P_T(p)$ be the closure, except for p, of

$$z \;\&\; \bigwedge_{1\le i\le n} F_i \;\&\; [Z(z) \rightarrow P_n(p,z)]$$

$$\&\; \bigwedge_{1\le i\le n}\;\bigwedge_{1\le j\le r_i+1} [E(u,v) \;\&\; P_i(x_1,\ldots,x_{j-1},u,x_{j+1},\ldots,x_{r_i+1}) \rightarrow P_i(x_1,\ldots,x_{j-1},v,x_{j+1},\ldots)].$$

4.4. If $P_T(p)$ means $p \in T$, then

$$F_T \asymp_{Df} \neg \forall x \, P_T(fx) \in \hat{\mathcal{L}}_{Pf}$$

means that the projection on T of the function g s.t. $f(x) = \bar{g}(x)$ is not infinite.
If, under Church's thesis, F_T is valid, then g ranges over all recursive functions, and hence every recursive branch of T is not infinite.
Since $\mathcal{F} =_{Df.} \{F_T \mid T \text{ is a p.r. binary tree}\}$ is structurally recognizable, we have the situation described in 4.1.

4.5. Using Jockusch [70] we show now that $\mathcal{F} \cap \mathcal{V}$ (the class of valid formulas of $\mathcal{F}$ under Church's thesis) is intuitionistically non-r.e.

Notation: $W_e \asymp$ the e^{th} r.e. set in some fixed enumeration.

$T_e \asymp$ the e^{th} p.r. binary tree in some fixed enumeration.

4.5.1. Lemma. There is a p.r. function h, s.t. the infinite branches of $T_{h(e_0,e_1)}$ are exactly the characteristic functions of the sets S s.t.

$$W_{e_0} \subset S \subset \sim W_{e_1} .$$

Proof. Define (using the notation of Kleene-Vesley [65])

$$T^{(e_0,e_1)}(x) \asymp_{Df} \text{Seq}(x) \ \& \ \forall(i<\text{lh}x)\ (x)_i < 2 \ \&$$

$$\forall(i<\text{lh}x)\ \{[\exists(z\leq x)\ T(e_0,i,z) \rightarrow (x)_i = 1$$

$$\& \ \exists(z\leq x)\ T(e_1,i,z) \rightarrow (x)_i = 0].$$

The first two conjuncts assure $T^{(e_0,e_1)}$ to be a set of binary sequence-numbers (the tree-property $x * n \in T \Longrightarrow x \in T$ is implied by the third conjunct).
The last main conjunct express that x cannot be denied to represent the initial sequence of an adequate S by computations of length $\leq$ x.

The bounds on all quantifiers assure $T^{(e_0,e_1)}$ to be p.r., and the s-m-n theorem implies the existence of the desired p.r. function h.

4.5.2. Lemma. There is a p.r. function k s.t.

$$W_e \text{ finite} \implies W_{k(e)} \text{ finite}$$
$$W_e \text{ infinite} \implies W_{k(e)} = \mathbb{N}$$

Proof. Define

$$W_{k(e)} =_{Df.} \{x \mid \exists (y \geq x) y \in W_e\}.$$

k is a p.r. function by the s-m-n theorem.

4.5.3. Lemma

$$\mathbb{F} =_{Df.} \{e \mid T_e \text{ has effectively no infinite recursive branch}\}$$

is (Σ_2^0-) completely-productive (and therefore not-r.e.).

Proof. Let A, B be fixed, disjoint, effectively-inseparable r.e. sets, and let

$$\mathbb{I} =_{Df.} \{e \mid W_e \text{ infinite}\}.$$

Define: $W_{g_0(e)} =_{Df.} A \cap W_{k(e)}$; $W_{g_1(e)} =_{Df.} B \cap W_{k(e)}$

$$f(e) =_{Df.} h(g_0(e), g_1(e)),$$

where k, h are as above, and f is p.r. (by the s-m-n theorem). By 4.5.1-2.

$$W_e \text{ finite} \implies W_{g_0(e)} \text{ and } W_{g_1(e)} \text{ are finite and disjoint}$$
$$\implies f(e) \notin \mathbb{F}.$$

$$W_e \text{ infinite} \implies W_{g_0(e)} = A \text{ and } W_{g_1(e)} = B$$
$$\implies f(e) \in \mathbb{F} \text{ effectively}.$$

Hence $\mathbb{I} \equiv_m \mathbb{F}$, and since $\mathbb{I}$ is (effectively-) productive (Rogers [67] §7.3 p.84 and §7.4), $\mathbb{F}$ is also (effectively-) productive (Rogers [67] §7.3 th. V(b)), and hence (effectively-) completely-productive.

4.6. The constructive non-r.e. of $\mathbb{F}$ (by complete-productivity) of 4.5.3 yields the desired intuitionistic result:
Let p be a (total) recursive functions for the complete-productivity of $\mathbb{F}$,

$$p(e) \in (W_e \sim \mathbb{F}) \cup (\mathbb{F} \sim W_e) \quad \text{or}$$

$$(*) \qquad \neg[p(e) \in W_e \;\&\; p(e) \in \mathbb{F}] \;\&\; \neg[p(e) \notin W_e \;\&\; p(e) \notin \mathbb{F}].$$

Intuitionistically

$$(**) \qquad p(e) \in W_e \equiv \exists x \exists y [T(e,x,y) \;\&\; p(e) = U(y)]$$

$$(***) \qquad p(e) \in \mathbb{F} \equiv \forall u \{\forall x \neg\forall z \neg T(u,x,z) \rightarrow$$

$$\neg\forall a\, [\forall(i<lha) \neg\forall y \neg[T(u,i,y) \;\&\; (a)_i = U(y)] \rightarrow T_{p(e)}(a)]\}$$

Substituting (**) and (***) in (*), the existential quantifier of (**) is everywhere negated in (*), therefore the classical provability of (*) implies its provability in intuitionistic recursion theory.
Now, if $\mathbb{F} = W_t$ for some t, then

$$p(t) \in W_t \implies \text{contradiction by the first conjunct of } (*);$$

$$p(t) \notin W_t \implies \text{contradiction by the second conjunct.}$$

4.7.1. The non-completeness result for If can be rewritten to represent non-<u>schematic</u>-completeness (with recursive functions as parameters) of I. Instead of writing $P_T(f(x)) \in \hat{L}_{Pf}$ we can write $P_T(f;x)$ with f as a a metamathematical parameter, which - in a model of z - is used by

$$P_T(x) \;\&\; \forall(i<lhx)\; f(i) = (x)_i,$$

where the interpretation of lhx and $(x)_i$ depends on that of E, Z and S. Schematic validity reads then

$$(\forall \text{rec}.f)\; \neg\forall x\; P_T(f;x).$$

4.7.2. This result can be considerably strengthen to more significant parameters of schema, e.g. the infinite proceeding sequences of the system CS of Kreisel-Troelstra [70]. If 'α' is a letter for lawless sequences, and 'a' - for lawlike functions, then schematic completeness of I with (CS-) ips's as parameters follows from the fact that in CS

$$(*) \qquad \forall\alpha F(\alpha) \longleftrightarrow \forall a F(a) \qquad \text{for negative } F;$$

and here $F(\alpha) \asymp \neg\forall x\, P_T(\alpha;x)$.

(*) is a straightforward consequence of $\forall\alpha\neg\neg\exists a\,[\alpha=a]$ (Kreisel-Troelstra [70] §6.2.4) which means that it is absurd to identify an arbitrary (lawless) ips as non-lawlike.

4.7.3. For the lawless sequences of Kreisel [58a], which obey fewer axioms than those of CS, the non-schematic-completeness follows already from Gödel's construction described in Kreisel [62], which shows that for α from the full binary spread weak schematic completeness implies

$$(G) \qquad \forall\alpha\, \neg\neg\exists n\, A(n,\alpha) \rightarrow \neg\neg\forall\alpha\exists n\, A(n,\alpha)$$

for every p.r. predicate A. But the lawless sequences of Kreisel [58a] demonstrately do not satisfy (G), since (remark 7.4 there) $\forall\alpha\neg\forall n\alpha(n) = 0$ but $\neg\forall\alpha\exists n\alpha(n) = 0$.

4.8. On the other hand the results above are obviously adaptable to extensions of I which are consistent with Church's thesis. This means that Markov's schema

$$(M) \qquad \forall x[Ax \vee \neg Ax] \;\&\; \neg\neg\exists x Ax \rightarrow \exists x Ax,$$

which is consistent with Church's thesis, does not imply completeness. Hence also

$$(G^+) \qquad \forall\alpha\, \neg\neg\, \exists n A(n,\alpha) \rightarrow \forall\alpha\exists n A(n,\alpha)$$

which is a special case of (M), does not imply weak-completeness, although implied by strong-completeness. (G) itself, however, does imply weak completeness (Dyson-Kreisel [61]), and is therefore equivalent to it.

Remarks

(1) In the language of propositional calculus the notion of plurality is absent. In L_{Pf} the function-symbols are also formal letters.

(2) Compare the following definition of Cantor: "A set is any collection into a whole of definite and separate objects". (Cantor: "Contribution to the Foundations of the Theory of Transfinite Numbers").

(3) If one attacks the classical notion of semantics in a non-formalistic way, there is no reason even then that the general notions of semantics can be described in set-theoretical terms without some essential restriction. The possibility to do it is in fact a corollary of the classical completeness theorem, as was pointed out by Kreisel [64] §1.83, and explained again in Kreisel [65].

(4) Kreisel does it in [62], although he mentions there indirectly the theory implicite in every model: "A is valid, i.e. for all species R there is a 'proof' of A_R". This distinction, which disappears in later writings of Kreisel does not play a role in the abstract theory of constructions developed in [62], since this theory is intended to deal with those "proofs" impredicatively. For a result in the direction of soundness there is no harm in this impredicativity (which is just a formalization of Heyting's explanation on the meaning of the logical symbols in intuitionistic mathematics). This, however, becomes an essential limitation when one touches completeness, i.e.: the problem of exhaustion of validity, which must involve the actual meaning of these proofs. As Church puts it ([56] fn. 143), to speak abstractly about semantics can belong (if we like) to "theoretical syntax", and semantics begins by fixing the meaning of particular interpretations.
Kreisel recognizes indeed indirectly the limits of the impredicative theory of [62] for actual semantical problems: "Undoubtly, the most important open problem is to give a foundation for the introduction of species by means of (generalized) inductive definitions, for which the corresponding principles of proof by induction can be derived".

(5) This distinction was pointed-out already by Kreisel [65] for classical metamathematics. In §2 there, 'Val α' stands for the "intuitive validity" of α (in fact, "an assertion on a logical basis"), to distinguish it from 'Vα' - "α is valid in all set-theoretical structures". It is the second notion whose problematics in the intuitionistic case we analyse.

(6) This is similar in a way to the principle of continuity in intuitionistic analysis. There, however, the modification applies to objects (functionals) specified by existential quantifiers, and results therefore in a strengthening of assertions; here, on the other hand, it applies to the range implicite in a universal quantifier, and it is a weakening. On limited validity - compare also Kreisel [58], p.318.

(7) A more formal aspect of the situation: the uncompleted objects cannot be incorporated in the description of a fixed model, since they do not satisfy condition (2) of 2.5 below.

(8) The terminology of (i) and (iii) is of Kreisel [58]. Kreisel mentions there also a uniform completeness

$$\mathbb{V}A\ \exists M\ [M \models A \implies \exists x\ \mathbb{P}rov(x,A)].$$

This is the kind of result of Hilbert-Bernays [39], which can be applied to fragments of I (cf. §4 below). However, it is different from (i)-(v). It is stronger <u>classically</u> than (i), and the formula meta-symbol 'A' must be quantified here (in (i)-(v) it is a parameter).

(9) Remark 2.3 of Kreisel [58] which disaccords with 2.1.1 seems to be erroneous: Since $Prov_I$ is an arithmetical predicate

$$\vdash_I \neg\forall x\ \neg Prov_I(x,\ulcorner A\urcorner) \Rightarrow\ \vdash_C \neg\forall x\ \neg Prov_I(x,\ulcorner A\urcorner),$$

and even the converse holds.

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